UNIVERSITY ENTRANCE EXAMINATION 2018

MATHEMATICS ('A' LEVEL EQUIVALENT)

Time allowed : 2 hours

<u>INSTRUCTIONS TO CANDIDATES</u>

1. This examination paper has TWO (2) sections – A and B, and comprises **SIXTEEN (16)** printed pages.

2. Attempt all sections.

3. Answer all questions in section A. Indicate your answers on the answer paper provided. Each question carries 2 marks. Marks will not be deducted for wrong answers.

4. Answer FOUR (4) questions from Section B with not more than THREE (3) from any one option. Write your answers on the answer paper provided. Begin each question on a fresh sheet of paper. Write the question number clearly. Each question carries 15 marks.

5. A non-programmable scientific calculator may be used. However, candidates should lay out systematically the various steps in the calculation.

6. At the end of the examination, attach the cover paper on top of your answer script. Complete the information required on the cover page and tie the papers together with the string provided. The colour of the cover paper for this examination is **GREEN**.

7. Do not take any paper, including the question paper and unused answer paper, out of the examination hall.

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NUS National University of Singapore

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SECTION A (40 Marks)

Answer all questions in this section. Each question carries 2 marks.

1. The inequality $25 - |5x| \geq |20x|$ has solution

   (A) $-5 \leq x \leq 5$

   (B) $0 \leq x \leq 5$

   (C) $0 \leq x \leq 1$

   (D) $-1 \leq x \leq 1$

   (E) none of the above

2. The graph of the function $f(x) = x^2 + \cos x + 2$ is symmetric about

   (A) the x-axis

   (B) the origin

   (C) the y-axis

   (D) the line $y = x$

   (E) none of the above

3. If

$$\sec x + \tan x = \frac{2}{3},$$

   then the value of $\sec x - \tan x$ is

   (A) $\frac{1}{3}$

   (B) $\frac{1}{2}$

   (C) 1

   (D) $\frac{3}{2}$

   (E) none of the above

4. ABC is a triangle in which $AB = 9\,\text{cm}$, $BC = 40\,\text{cm}$ and $AC = 41\,\text{cm}$. The area of triangle ABC is

(A) $180\,\text{cm}^2$

(B) $184.5\,\text{cm}^2$

(C) $360\,\text{cm}^2$

(D) $820\,\text{cm}^2$

(E) none of the above

5. The derivative of

$$\ln(\ln\sqrt{x}\,)$$

with respect to x is

(A) $\dfrac{2}{x\ln x}$

(B) $\dfrac{1}{x\ln x}$

(C) $\dfrac{1}{2x\ln x}$

(D) $\dfrac{1}{4x\ln x}$

(E) none of the above

6. Water is poured into a cylindrical container of radius $6\,\text{cm}$ at the rate of $6\pi\,\text{cm}^3\text{s}^{-1}$. The rate at which the water level is rising is

(A) $\frac{1}{6}\,\text{cm}\,\text{s}^{-1}$

(B) $\frac{1}{3}\,\text{cm}\,\text{s}^{-1}$

(C) $3\,\text{cm}\,\text{s}^{-1}$

(D) $6\,\text{cm}\,\text{s}^{-1}$

(E) none of the above

7. The vector $\begin{pmatrix} \frac{4}{5} \\ v \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} w \\ \frac{3}{5} \end{pmatrix}$. The value of $\dfrac{v+w}{v+2w}$ is

(A) $-\frac{1}{2}$

(B) $-\frac{1}{5}$

(C) $\frac{1}{5}$

(D) $\frac{2}{5}$

(E) none of the above

8. A geometric progression P has first term 2 and sum to infinity 6. The sum to infinity of the new geometric progression obtained by squaring every term in P is

(A) $\frac{9}{5}$

(B) $\frac{18}{5}$

(C) $\frac{81}{5}$

(D) $\frac{162}{5}$

(E) none of the above

9. The area of the largest rectangle that can be inscribed in a circle of radius $20\,\text{cm}$ is

(A) $400\,\text{cm}^2$

(B) $800\,\text{cm}^2$

(C) $1200\,\text{cm}^2$

(D) $1600\,\text{cm}^2$

(E) none of the above

10. Two fair dice are thrown and the score on each die is noted. The probability that the total scores on both dice is odd is

(A) $\frac{9}{36}$

(B) $\frac{12}{36}$

(C) $\frac{18}{36}$

(D) $\frac{24}{36}$

(E) none of the above

11. Which of the following is the result of completing the square of the expression $-20x^2 + 6x - 1$?

(A) $-20\left(x - \dfrac{3}{20}\right)^2 - \dfrac{9}{20}$

(B) $-20\left(x - \dfrac{3}{20}\right)^2 - \dfrac{11}{20}$

(C) $-20\left(x - \dfrac{3}{20}\right)^2 - \dfrac{9}{200}$

(D) $-20\left(x - \dfrac{3}{20}\right)^2 + \dfrac{11}{200}$

(E) none of the above

12. The equation of a curve is $y = 8x^3 - 12x^2 + 1$. The value of c for which the line $6y = c + 4x$ is a normal to the curve is

(A) -4

(B) -2

(C) 2

(D) 4

(E) none of the above

13. The function $f(x) = 2ax + b$ is such that $f(\frac{1}{2}) = 4$ and $f^{-1}(5) = \frac{3}{2}$. The value of b is

(A) $\frac{5}{2}$

(B) $\frac{7}{2}$

(C) $\frac{9}{2}$

(D) $-\frac{5}{2}$

(E) none of the above

14. The points P, Q and R have coordinates $(1, 4)$, (a, b) and $(4, 13)$ respectively. Suppose P, Q and R are collinear. Then the value of $\dfrac{b-13}{a-4}$ is

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) 3

(D) 4

(E) none of the above

15. The number of ways in which the letters of the word *ISOSCELES* can be arranged so that the first letter is a vowel is

(A) 5040

(B) 6720

(C) 10080

(D) 13440

(E) none of the above

16. Which option corresponds to the partial fraction decomposition of the rational function $-\dfrac{13}{150x^2 - 25x - 6}$?

(A) $-\dfrac{2}{10x-3} + \dfrac{3}{15x+2}$

(B) $\dfrac{2}{10x-3} + \dfrac{3}{15x+2}$

(C) $\dfrac{2}{10x-3} - \dfrac{3}{15x+2}$

(D) $-\dfrac{2}{10x-3} - \dfrac{3}{15x+2}$

(E) none of the above

17. The number of ways to choose a pair of distinct numbers a and b from the set $\{41, 42, \ldots, 91\}$ such that $\left|a - b\right| \leq 3$ is

(A) 141

(B) 144

(C) 147

(D) 150

(E) none of the above

18. The maximum value of the function $f(x) = (3\cos x - 4)^2 - 1$ is

(A) 8

(B) 15

(C) 24

(D) 48

(E) none of the above

19. The derivative of $\dfrac{\ln(5-x)}{e^{5x}}$ with respect to x is

(A) $e^{-5x}\left(\dfrac{1}{x-5} - 5\ln(5-x)\right)$

(B) $e^{-5x}\left(\dfrac{5}{x-5} - 5\ln(5-x)\right)$

(C) $e^{-5x}\left(\dfrac{5}{x-5} - \ln(5-x)\right)$

(D) $e^{-5x}\left(\dfrac{1}{x-5} + 5\ln(5-x)\right)$

(E) none of the above

20. Suppose $-9 \leq x \leq 7$ and $-6 \leq y \leq 8$. Then the largest value of $(y-3)^2 + (x-2)^2$ is

(A) 80

(B) 180

(C) 185

(D) 208

(E) none of the above

SECTION B (60 Marks)

Answer FOUR (4) questions with not more than THREE (3) from any one option.

Option (a) - Pure Mathematics

21(a). Find the integral $\int 2\sin^{-1} x \, dx$. [5 Marks]

21(b). Find the integral $\int \frac{\sin 4\theta}{\sin 2\theta \sin^2\theta} \, d\theta$. [5 Marks]

21(c). Evaluate $\int_0^{\frac{\pi}{4}} \frac{3\sin^2 x + 5\tan x}{6\cos^2 x} \, dx$. [5 Marks]

22(a). Solve the following inequalities

(i) $|4x+1| > |2x-3|$, [4 Marks]

(ii) $\frac{6}{|x+2|+1} < |x+2|$. [6 Marks]

22(b). Solve the equation

$$81z^4 + 9z^2 + 1 = 0,$$

giving the roots in the form $a + bi$, where a and b are real numbers. [5 Marks]

23(a). Find the general solution of the following differential equation

$$(4+e^x)\frac{dy}{dx}+e^x y=0,$$

expressing y in terms of x.

Sketch the solution curve for which $y=\frac{1}{5}$ when $x=0$. [8 Marks]

23(b). By using the substitution $y=v+x$, find the general solution of the differential equation

$$x\frac{dy}{dx}=3y-2x,$$

expressing y in terms of x. Sketch the solution curve which passes through the point $(-1,1)$. [7 Marks]

24. The planes π_1 and π_2 have equations

$$3x+2y+z=4 \quad \text{and} \quad 2x+3y+z=5$$

respectively. The point A has coordinates $(2,2,1)$. The plane π_3 containing the point A is perpendicular to each of the planes π_1 and π_2.

(i) Find the shortest distance from the point A to the plane π_1. [3 Marks]

(ii) Find the cosine of the angle between the planes π_1 and π_2. [3 Marks]

(iii) Find a cartesian equation for the plane π_3. [3 Marks]

(iv) Find a vector equation for the line of intersection L of the planes π_1 and π_2. [2 Marks]

(v) Find the point N on line L which is nearest to the point A. [4 Marks]

Option (b) - Particle Mechanics

[In this section, take the acceleration due to gravity to be 9.8 ms^{-2}, unless otherwise stated. Give non-exact numerical answers correct to three significant figures, unless otherwise specified.]

25. A cyclist and her bicycle have a total mass of m kg. When she free-wheels down slopes inclined at angles $\sin^{-1}\left(\frac{1}{10}\right)$ and $\sin^{-1}\left(\frac{1}{20}\right)$ to the horizontal, she does so with steady speeds V ms^{-1} and $\frac{1}{4}V$ ms^{-1} respectively. The resistance to motion that she experiences is $a + bv$ N when she is travelling at a speed of v ms^{-1}, where a and b are constants. Show that $a = \dfrac{mg}{30}$, where g is the acceleration due to gravity. Express b in terms of m, g and V. [5 Marks]

    Find also the rate P at which she works when she cycles on a level ground at a constant speed of $\frac{1}{2}V$. Leave your answer in terms of m, g and V. [3 Marks]

    She now cycles along a straight horizontal road with the engine switched off. Assuming that she starts off with a speed V ms^{-1}, and she still experiences the same resistance of $a + bv$ N when her speed is v ms^{-1}, show that she will come to rest after travelling a distance of $\dfrac{15V^2}{2g}\Big(2 - \ln 3\Big)$. [7 Marks]

26. A particle of mass m kg is projected under gravity with horizontal and vertical components of velocity U ms^{-1} and V ms^{-1} respectively. Motion takes place in a medium, which only produces a horizontal resistance of magnitude mku N, where k is a constant and u ms^{-1} is the horizontal component of the velocity of the particle at time t seconds. Show that the horizontal range R metres from the point of projection is given by

$$R = \frac{U(1 - e^{-kT})}{k},$$

    where $T = \dfrac{2V}{g}$ and g is the acceleration due to gravity. [7 Marks]

    Find the horizontal distance D metres travelled by the particle before reaching the highest point of its path. [4 Marks]

    Hence, show that $D > \dfrac{R}{2}$. [4 Marks]

27. Two particles A and B, of masses m kg and $2m$ kg respectively, are attached to the ends of a light inextensible string which passes over a smooth fixed pulley. The particles are released from rest with the parts of the string on each side of the pulley hanging vertically. When particle B has moved a distance h metres, it hits on the horizontal ground and does not rebound. Give your answers in terms of m, g and h, where g is the acceleration due to gravity, for the following:

(i) Find the speed of the particles just before B hits the ground. [3 Marks]

(ii) Find the direction and magnitude of the impulse that the particle B exerting on the ground. [2 Marks]

(iii) In the subsequent motion, B will rest on the ground until it is jerked into motion again. Find the period of time when B is resting on the ground before it is first jerked into motion. [3 Marks]

(iv) Find the speed when B first leaves the ground. [2 Marks]

(v) After jerked into motion, B rises up and then drops to hit the ground again. Find the distance B travels between the first hit and the second hit on the ground. [2 Marks]

(vi) The motion continues with B hitting the ground and jerked into motion indefinitely. Find the total distance travelled by B from the instance that the particles are released from rest. [3 Marks]

28. A particle of mass m kg rests on a plane inclined at an angle α to the horizontal and the coefficient of friction between the particle and the plane is μ, where $\mu < \tan\alpha$. The particle is fastened to one end of a spring of natural length λ metres and modulus of elasticity λ N. The other end of the spring is fastened higher up the plane so that the spring lies along a line of greatest slope.

The particle is released from rest with the spring at its natural length. By writing down the energy equation for the motion, or otherwise, find an expression for the velocity $v\,\mathrm{ms}^{-1}$ of the particle when it has slid a distance x metres down the slope.

[6 Marks]

Find the distance the particle will slide down the slope before coming to rest.

[3 Marks]

Show that the particle will remain at rest in this position if $\frac{1}{3}\tan\alpha \leq \mu < \tan\alpha$.

[6 Marks]

Option (c) - Probability and Statistics

[In this section, probabilities should be expressed as either fractions in lowest terms or decimals with three significant figures.]

29. [In this question, give each of your answers as an exact fraction in its lowest term.]

Two fair dice are such that die A has 4 red faces and 2 green faces, whereas die B has 2 red faces and 4 green faces. A fair coin is flipped once. If it lands on a head, die A is tossed; otherwise, die B is tossed. Subsequently, the same die is tossed two more times.

(i) Find the probability of obtaining a red face in the first toss. [2 Marks]

(ii) Find the probability that red faces turn up in the first two tosses. [3 Marks]

(iii) What is the probability that no two consecutive green faces occur in the three tosses? [4 Marks]

(iv) If the first two tosses result in two red faces, find the probability that

(a) a red face is obtained at the third toss; [3 Marks]

(b) it is die A that is being tossed. [3 Marks]

30. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} kx(4-x), & 1 \leq x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

where k is a constant.

(i) Find the value of k. [3 Marks]

(ii) Sketch the graph of $y = f(x)$ and hence state the value of $E(X)$. Find $Var(X)$. [4 Marks]

(iii) The random variable W is defined by $W = 3 - 2X$. Find the median value of W. [3 Marks]

(iv) Find $E(W)$ and $Var(W)$. [2 Marks]

(v) A random sample consists of 60 independent observations of W is taken. Find an approximate value of the probability that the sample mean exceeds -0.6. [3 Marks]

31. The number of characters mistyped by a typist per page in the first draft of a publication has a Poisson distribution with mean 1.3. The number of mistyped characters on a page is independent of all other pages.

(i) Calculate the probability that a randomly chosen page contains exactly 2 mistyped characters. [2 Marks]

(ii) Calculate the probability that the first 10 pages contain at most 2 errors. [3 Marks]

(iii) Given that the probability of having no errors on each of n randomly chosen page is at least 0.01, find the greatest value of n. [3 Marks]

(iv) If 50 pages are randomly chosen, find the probability that

(a) there are at least 15 pages with exactly 2 mistyped characters. [4 Marks]

(b) the mean number of mistyped characters per page is at most one. [3 Marks]

32. The weights of boys in a certain junior college are normally distributed with mean 60 kg and standard deviation 7 kg. The weights of girls in this junior college are also normally distributed with mean 45 kg and standard deviation 10 kg.

(i) Find the probability that the weight of a boy in the college is less than 63 kg. [3 Marks]

(ii) Find the probability that the average weight of 5 boys in the college is greater than 65 kg. [4 Matks]

(iii) Find the probability that the total weight of 4 boys in the college exceeds that of 4 girls. [4 Marks]

A sample of 100 boys are selected at random from the college. Estimate the probability that less than 70 and more than 60 of the boys in this sample have weights less than 63 kg. [4 Marks]

END OF PAPER