1 Abstract

A system of three spin- \( \frac{1}{2} \) atoms allows the construction of a qubit - in the subspace of total angular momentum \( j = \frac{1}{2} \) that is not affected by any magnetic field activity on the magnetic moments of the atoms, provided that all three atoms experience the same magnetic field. If, however, there are stray fields of different direction or strength at the sites of the atoms, the qubit will slowly decohere. It is the objective of this study to examine the decoherence process and to establish the conditions, under which the lifetime of the qubit is sufficiently long for practical uses.

2 Introduction

2.1 Construction of the decoherence-free qubit

To construct the reference-frame-free (rff) qubit one uses the ladder operator \( J = J_x - iJ_y \) for the whole three-atom system and defines its two orthogonal lowering partner operators

\[
\Omega_-(\lambda) = \frac{1}{\sqrt{3}} \sum_{l=1}^{3} \omega^\lambda \sigma_-^{\lambda}(l))
\]  

(2.1)

where \( \omega = e^{\frac{2\pi i}{3}} \), \( \sigma_-^{\lambda}(l) \) refers to the lowering operator for only the atom at site \( l \) and \( \lambda = 1, 2 \) labels the state of the rff qubit. For our calculations we usually use the following basis in the total angular momentum \( j = \frac{1}{2} \) subspace:

\[
|j = \frac{1}{2}, m, \lambda > = \Omega_-(\lambda)J_z^{1-m}|j = \frac{3}{2}, m = \frac{3}{2} >,
\]  

(2.2)

where \( m \) indicates the \( z \)-component. For the \( j = \frac{3}{2} \) we use the ordinary \( |j, m > \) basis. For a more detailed discussion of this construction see [1].

2.2 Perturbation of setup

In this UROPS project we study the stability of the rff qubit under the time evolution given by a spatially constant field \( B \) along the \( z \)-axis acting on all 3 atoms with Hamiltonian
\[ H_0 = \vec{B} \cdot \sum_i \vec{\sigma}_i \]  

(2.3)

and stray fields \( b_i(t) \) where \( i \in \{1, 2, 3\} \) acting only on the atom with index \( i \). In the following we set \( \mu_B = 1 \) and \( \hbar = 1 \) for easier calculations. The time evolution due to stray fields is described by Hamiltonian

\[ H_1 = \sum_i \vec{b}_i \cdot \vec{\sigma}_i. \]  

(2.4)

In the rfi-subspace all density matrices commute with \( H_0 \) as they are rotationally invariant states. The total hamiltonian is given by \( H = H_0 + H_1 \).

3 Results

3.1 Evolution of rff-qubit under constant perturbation

As a first approach consider all fields to be constant in time. The effect on the rff qubit by a stray field orthogonal to the \( z \)-axis can get stabilized very well by a large overall field \( B \) along the \( z \)-axis. However, if the stray field is parallel to the overall field \( B \), the overall field has no stabilizing effect at all, as in this situation \( H_0 \) and \( H_1 \) commute. This result can be seen in the following graphics, where as an example the time evolution of the projection of state \( Q_{11}/2 \) (\( Q_{11} \) is defined as \( |\frac{1}{2}, \lambda = 1 > < \frac{1}{2}, \lambda = 1 | \otimes 1 \) on its initial state (i.e. \( < Q_{11}(t)/2, Q_{11}(0)/2 > \)) is plotted. The \( x \)-axis of this plot corresponds to the strength of the stabilizing field \( B \) and the \( y \)-axis corresponds to time.

In the plot we see that with increasing \( B \) the effect of a stray field orthogonal to \( B \) gets reduced drastically.

3.2 Evolution of rff-qubit with fluctuating perturbation

Consider the Hamiltonian with \( B_0 \) and \( b \) in units of energy:

\[ H_1(t) = B_0 \sigma_z + b(\sigma_{lx} \xi_{lx}(t) + \sigma_{ly} \xi_{ly}(t) + \sigma_{lz} \xi_{lz}(t)) \]  

(3.1)

The random variables \( \xi \) have space and time correlation and represent the noise. The spacial correlation is a result from modeling the random \( b \)-fields according to Maxwell’s equations. In our setup the atoms will be trapped close together by a laser so we can suppose that the field varies spacially only in first order. The simulation of this first order approximation for a equilateral triangular setup with distance 1 between atoms in the xy-plane is done for different time correlations below.
3.3 Simulation with white noise

Consider the case $\gamma \to \infty$ i.e. white noise. For the simulation we use the approximation (bars indicate averaging over the probability distribution)

$$\rho(t + dt) = \rho(t) - i[H_0, \rho]dt + \frac{1}{2} H_1^2 \rho + \frac{1}{2} \frac{1}{2} \rho H_1^2 dt$$  \hspace{1cm} (3.2)

which is a Lindblad form master equation and can be solved analytically using the corresponding Lindblad superoperator $L$:

$$\rho(t) = e^{Lt} \rho(0)$$  \hspace{1cm} (3.3)

If $H_0 = 0$, the resulting operator $L$ has only real Eigenvalues and all of them are smaller than or equal to 0. Thus the state decays in first order and application of Zeno effect is not possible according to the white noise model. There is exactly one eigenvalue equal to zero which has the stationary state represented by the eight dimensional unit matrix as eigenvector. All other eigenstates decay with rates ranging from about $-5$ to $-24$. We are especially interested in states used for encoding the r-qubit. The calculations predict that the z-component of the rff state is the most unstable one compared to x- and y-components which is in agreement with an analytical calculation by Han Rui.

3.4 Simulation with colored noise

For small correlation times compared to the random field strength one can use an integration scheme from [3] to simulate the noise. The time evolution of the quantum state can then be done using the exact unitary transformation at each timestep. Thus the error in this calculation occurs only because of the noise simulation.

Using this model quantum states will decohere only in second order. Thus Zeno effect can be applied to suppress transitions between states of different total angular momentum $j$ by continuously monitoring the $j$ value via measurement in the lab. To simulate the time evolution of the density matrix $\rho$ one has to average over many runs.

Below is a plot of a Zeno-slowed down decohering rff-state. The graph has two curves which represent different runs of the numerical simulation. The peaks occur at the time when the measurement is done.

![Plot of Zeno-slowed down decohering rff-state.](image)

**time development of $|\frac{1}{2}, -\frac{1}{2}, 2\rangle$ with measurements of $j$**

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The corresponding time evolution without measurements would have decohered to \( \frac{1}{2} \) already.

By increasing the constant B-field in z-direction one can also suppress transitions between different values of \( m \).

### 3.5 Spin-spin interaction

In the above models spin-spin interaction between the atoms is ignored. We now include this interaction which is of the form:

\[
H_{AB} = \frac{1}{R_{AB}^3} \left[ \mu^A \cdot \mu^B - 3(\mu^A \cdot \hat{R}_{AB})(\hat{R}_{AB} \cdot \mu^B) \right]. \tag{3.4}
\]

Where \( AB \) refers to different atoms, \( R_{AB} \) is the distance between them and \( \hat{R}_{AB} \) is the unit vector pointing from one atom to the other. For \( g_s = 2 \), a constant field in z-direction and a setup of the three atoms sitting in an equilateral triangle of distance 1 in the xy-plane we get the following result: In our preferred basis four states are eigenstates already and the remaining four by four matrix can be decomposed into two two-dimensional subspaces with no transitions between them. In each subspace the transition rate \( \gamma = 3\sqrt{3} \). The time evolution for arbitrary two-dimensional quantum systems of this form is given by Rabi's formula [2]:

\[
\text{where } \omega = \frac{E_1 - E_2}{\hbar} \text{ and } c_2 \text{ is the probability to be in state 2 after time } t \text{ when starting in state 1. For a large energy gap } \omega >> \gamma \text{ the transition is suppressed. In our case we can achieve this with a very strong } B_z \text{ as it enlarges the energy gap. Without an external field the transition is strong as the off-diagonal elements are big compared to the diagonal elements. The maximal value of } |c_2|^2 \text{ is } \frac{27}{272} \approx 1 \text{ for both transitions in that case.}
\]

In the final setup one has to find a compromise for the distance between the atoms: If it is too large, the field will not be homogeneous enough, if it is too small the spin-spin interaction will perturb the state.

### 4 Conclusion

Various techniques can be used to slow down decoherence of the qubit in different setups. Above calculations will be adjusted to the planned experiment.

**Literatur**

