On the Non-trivial Zeta Zeroes and its significance in Quantum Systems

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Abstract
Motivated by Hilbert and Pólya, there were many developments in the field of spectral theory related to the study of Riemann’s Zeta zeroes. In this paper, Riemann’s 1859 paper is discussed, deriving the formula for the prime counting function that Riemann obtained. Another formula due to Weil will also be derived. The line of development of classical mechanics to quantum mechanics were presented and Floquet operator of periodic Hamiltonian was discussed and used in the kicked rotator model. A brief layout of using trace Green’s function to determine the density of state was carried out and the Gutzwiller’s trace formula was quoted.

Functional Equation of \( \zeta \)

The zeta function, \( \zeta(s) \) defined for \( Re(s) > 1 \) is given by

\[
\zeta(s) = \sum_{n} \frac{1}{n^s}
\]

(1)

Using theta function defined over right half-plane,

\[
\psi(s) = \sum_{n=1}^{\infty} e^{-\pi n^2 s}
\]

(2)

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Define $\xi(s)$ such that

$$
\xi(s) = (s - 1)\pi^{-\frac{s}{2}}\Pi\left(\frac{s}{2}\right)\zeta(s)
$$

(3)

The functional equation for

$$
\xi(s) = \xi(1 - s)
$$

(4)

can be obtained using Mellin’s transform.

**Prime Counting Function**

The use of Mellin’s transform on

$$
\log \zeta(s) = s \int_0^\infty x^{-s-1}J(x)dx
$$

results in

$$
J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\log \zeta(s)}{s} x^s ds
$$

(6)

Define $Li(x)$ as the Cauchy principal value of the logarithmic integral, then a formula for $J(x)$ is obtained

$$
J(x) = Li(x) - \sum_{\text{Im}(\alpha)>0} \left[ Li(x^\alpha) + Li(x^{1-\alpha}) \right] \\
+ \int_x^\infty \frac{1}{t(t^2-1)\log t} \, dt + \log \xi(0)
$$

(7)

Using the Möbius Inversion Theorem

$$
\pi(x) \approx \sum_{n=1}^\infty \frac{\mu(n)}{n} J(x^{\frac{1}{n}})
$$

(8)

allows us to compute the number of primes less than $x$ to a high accuracy. Using similar methodology, another formula due to Weil can be worked out.

Consider a compactly supported even function $g(u) \in C_c^\infty(\mathbb{R})$ such that

$$
h(r) = \int_{-\infty}^{\infty} g(u)e^{iru} \, du
$$

is an entire function and has exponential decay with large real $r$.

$$
\zeta^*(s) = \Psi(s)\zeta(s) = \pi^{-\frac{s}{2}}\Pi\left(\frac{s}{2} - 1\right)\zeta(s)
$$
then it can be worked out that
\[
    h(1) + h(0) - \sum_{\alpha} h(\alpha) = \sum_{p \in \mathbb{P}} \sum_{n} \frac{\log p}{p^2} (g(\log p^m) + g(-\log p^m))
\]
\[
    - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) Re \frac{\Psi'}{\Psi} \left( \frac{1}{2} + ir \right) dr
\]
(10)
where $\alpha$ are the non-trivial zeroes.

**Kicked Rotator**

We consider the chaotic kicked rotator model described by the following system of equations with parameter $k$
\[
    \theta_{n+1} = \theta_n + L_{n+1} \tag{11}
\]
\[
    L_{n+1} = L_n + k \sin \theta_n \tag{12}
\]

Define an averaging function $f_N(\Delta L) = \langle \delta(\Delta L - L_N + L_0) \rangle$. Under the assumption that the sequence $\{L_N\}$ are independent of each other, then
\[
    f_N(\Delta L) = \frac{1}{k\sqrt{N\pi}} e^{-\frac{\Delta L^2}{2Nk^2}} \tag{13}
\]

From which, it was worked out that the kicked rotator has a diffusion constant of
\[
    D_{\text{diff}} = \frac{1}{2} k^2 \tag{14}
\]

**Gutzwiller’s Trace Formula**

The central result in semiclassical approximation of the density of state of a quantum system is given by
\[
    d(E) = \frac{1}{\pi \hbar} \sum_{r} \frac{(T_P)_r}{||M_r - 1||^2} \cos \left[ \frac{1}{\hbar} S_r(E) - \frac{i\mu_r \pi}{2} \right]
\]
(15)
which it can be written as $d(E) = d_0(E) + d_{osc}(E)$ similar to that of the smooth part and oscillatory part of the Weil’s formula. Maslov index $\nu_r = \mu_r + a_r$ was used in the formula. With some manipulation, it can be worked out that
\[
    d_{osc}(k) = -\frac{1}{\pi} \text{Im} \left[ \frac{d}{dk} \ln Z(k) \right]
\]
(16)
where
\[ Z(k) = \prod_{p,m} \left\{ 1 - e^{i[kl_p - \frac{2\pi m}{2} - (m+\frac{1}{2})a_p]} \right\} \] (17)

If we express the Riemann zeta function as \( \zeta \left( \frac{1}{2} - ik \right) \) and using Golden key with \( p = e^{\ell_p} \), we obtain
\[ \zeta \left( \frac{1}{2} - ik \right) = \prod_p \left[ 1 - e^{ikl_p - \frac{1}{2}l_p} \right]^{-1} \] (18)

**Conclusion**

It will be of interest to investigate further connections which may ultimately prove the Riemann Hypothesis with the zeta zeroes as the spectrum of a hypothetical system.

**References**


