Entanglement in Many Body Systems

A.Z. Chaudhry\textsuperscript{1}, D. Kaszlikowski\textsuperscript{2}

Department of Physics, Faculty of Science, National University of Singapore 10 Kent Ridge Road, Singapore 117546

1 Abstract

This project studies entanglement in spin chains using the Heisenberg XX (or XY) model. Like the quantum harmonic oscillator, the Heisenberg spin chain is one of the fundamental models of physics. It is simply formulated, it describes real systems, such as ferromagnetism, which are experimentally accessible, and it has a rich and elegant mathematical structure. We first study and review the XX model in detail. In light of the growing importance of the study of entanglement in large many-body systems, we then construct an entanglement witness based on N-point correlation functions. We then apply this witness to the XX model for various number of particles under different temperatures and applied magnetic fields.

2 Heisenberg XX Model

We first review the Heisenberg XX model, which has been extensively studied. The one dimensional spin-1/2 Heisenberg XX model has the Hamiltonian

\[ H = J' \sum_{k=1}^{N-1} [S_k^x S_{k+1}^x + S_k^y S_{k+1}^y]. \]  

The approach that is adopted in solving this problem is to use the similarity between spin-1/2 raising/lowering operators and fermion creation/annihilation operators.

\[ a_k^\dagger = \frac{1}{2}(\sigma_k^x + i\sigma_k^y), \quad a_k = \frac{1}{2}(\sigma_k^x - i\sigma_k^y) \]  

This is followed by Jordan-Wigner transformations.

\[ c_k = \exp(\pi i \sum_{j=1}^{k-1} a_j^\dagger a_j) a_k, \quad c_k^\dagger = a_k^\dagger \exp(-\pi i \sum_{j=1}^{k-1} a_j^\dagger a_j) \]  

\textsuperscript{1}Student

\textsuperscript{2}Assistant Professor, Supervisor
We then perform another transformation.

\[ d_k^\dagger = \sum_{l=1}^{N} \alpha_{kl} c_l^\dagger, \quad d_k = \sum_{l=1}^{N} \alpha_{kl} c_l \tag{4} \]

Using these, the Hamiltonian is eventually cast in diagonal form

\[ H = J \sum_{k=1}^{N} E_k d_k^\dagger d_k \tag{5} \]

where \( J = J' h^2 / 2 \), \( E_k = \cos(\frac{k\pi}{N+1}) \) and \( \alpha_{kl} = \sqrt{2(N+1) \sin(\frac{k\pi}{N+1})} \).

Let us define a state \( |\Omega\rangle \) such that the \( d_k \) operator annihilates it. Clearly, this is one of the eigenstates of the Hamiltonian with eigenvalue 0. We find that, in general, \( d_{i_1}^\dagger d_{i_2}^\dagger \ldots d_{i_n}^\dagger |\Omega\rangle \) is an eigenstate of \( H \) with eigenvalue \( J(E_{i_1} + E_{i_2} + \ldots + E_{i_n}) \). Physically, \( |\Omega\rangle \) represents the state with all spins down. The state \( d_k^\dagger |\Omega\rangle \) is a spin wave.

The ground state is then studied in detail. The energy eigenstates are found to be eigenstates of the operator \( R = \sigma^x \otimes N \) as well. Therefore, the energy eigenstates can be written in terms of generalized GHZ states.

### 3 Deriving entanglement witness

Using these generalized GHZ states, we then construct an entanglement witness to detect entanglement. Entanglement is the phenomenon whereby the state of a system cannot be expressed as a product of the states of its subsystems. It is essential to develop techniques for entanglement detection in many-qubit systems. The most straightforward approach is via the so-called entanglement witnesses. Simply put, an entanglement witness is a Hermitian operator which can be constructed so that if the mean value of a state with the witness is greater than one, then we know that the state is entangled. In principle, the entanglement witness approach can be experimentally implemented.

Consider the Hermitian operator

\[ W = \frac{1}{2} \sum_{k \in \{0,1\}^N} b_k(Q_k^+ - Q_k^-) \tag{6} \]

The operators \( Q_k^\pm = \ket{G_k^\pm} \bra{G_k^\pm} \) are orthogonal projectors on the generalized GHZ states \( \ket{G_k^\pm} = \frac{1}{\sqrt{2}} (|\vec{k}\rangle \pm \sigma^x \otimes N |\vec{k}\rangle) \).

\(^3|\vec{k}\rangle \) is a string of zeros and ones
We find that the family of operators $\hat{U}W\hat{U}^\dagger$, where $\hat{U} = \prod_{n=1}^{N} U^{(n)}$ and $U^{(l)}$ is an arbitrary $SU(2)$ transformation on qubit $l$, becomes an entanglement witness if

$$\sum_{\vec{k}} |b_{\vec{k}}| \leq 2^N \quad (7)$$

If this condition holds, we have

$$\langle \psi_{\text{sep}} | \hat{U}W\hat{U}^\dagger | \psi_{\text{sep}} \rangle \leq 1 \quad (8)$$

for an arbitrary and fully separable state.

It can be shown that the trace of $W_U^4$ with an arbitrary density operator $\rho$ reads

$$A = \text{Tr}(W_U \rho) = \frac{1}{2^N} \sum_{\vec{k}} b_{\vec{k}} \lambda_{\vec{k}} \quad (9)$$

where

$$\lambda_{\vec{k}} = \sum_{\vec{l}} (-1)^{\vec{k}\cdot\vec{l}} \cos(\frac{\pi}{2} |\vec{l}|) T_{\vec{l}} \quad (10)$$

$$T_{\vec{l}} = \text{Tr}(U \sigma_{\vec{l}} U^\dagger \rho) \quad (11)$$

Within this family, the one that is best at detecting entanglement is the one which chooses the $b_{\vec{k}}$ to maximize $A$. This is simply given by

$$A = \max_{U}(|\lambda_{\vec{k_0}}|) \quad (12)$$

where $\lambda_{\vec{k_0}} = \max_{\vec{k}}(\lambda_{\vec{k}})$. A sufficient condition for $\rho$ to be entangled is therefore $A > 1$. However, it may be quite difficult to calculate $A$, but we can derive a useful lower bound $B$, i.e. $A \geq B$, which also witnesses entanglement. To derive $B$, put $b_{\vec{k}} = 2^N \lambda_{\vec{k}} (\sum_{\vec{l}} \lambda_{\vec{l}})^{-1}$. We get

$$A \geq B = \max_{U} \left( \sum_{\vec{l}\text{even}} T_{\vec{l}}^2 \right) \quad (13)$$

So if $B > 1$, we can say that the state is entangled. This is our fundamental result.

$^4W_U = \hat{U}W\hat{U}^\dagger$
4 Results and Discussion

We now apply the entanglement witness that we have to try and detect entanglement in the XY model for different number of particles. We allow different temperatures and different magnetic fields. We find that as temperature decreases, entanglement should increase as thermal fluctuations become less and less. We also find that if we increase the applied magnetic field beyond some critical value, we will not be sure whether or not there is entanglement. An applied magnetic field introduces some disturbance in the system, so it is logical that as we increase the magnetic field, entanglement should become less likely.

We can also witness different types of multipartite entanglement. After all, if we find that eight qubits are entangled, it could be that only two of them are in an entangled state; this entangled state and the state of the other six qubits may be separable. So, in this way, it is possible to have different types of multipartite entanglement.

We find that the onset of at least 2-party entanglement (which is of course just the onset of entanglement) is at a higher temperature for $N = 4$ compared to $N = 8$, as before. We cannot say whether or not there is at least 3-party entanglement in both cases.

What we have done is theoretically applicable to any number of particles; however, computationally the problem that we are now faced with is that the approaches we have considered become intractable for a large number of particles. Indeed, we cannot perform calculations even for $N = 12$. It would be worthwhile to find ways to do the calculations for a larger number of particles.

References