Functions of Bounded Variation

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ABSTRACT

Functions of bounded variation is a special class of functions with finite variation over an interval. Throughout this paper, we study the behaviour of these functions and derive a strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$ through the use of functions of bounded variations. We begin by discussing some of the basic properties of this class of functions. Then, we look into the space of these functions and its norm, and relate them to continuity, differentiability and integrability. Next, we cover some applications of functions of bounded variations in Riemann-Stieltjes integration. We also discuss the conditions required for Riemann-Stieltjes integrals to exist, some of its properties and its similarities to Riemann integrals. Finally, we introduce Fourier transforms and the various properties of the Fourier transformation. Together with the properties of functions of bounded variations and Riemann-Stieltjes integrals, we prove the strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$.

PROPERTIES OF FUNCTIONS OF BOUNDED VARIATION

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function and let $\Gamma = \{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a, b]$. Define

\[ p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+ \]  \hfill (1)

\[ n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^- \]  \hfill (2)

\[ t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \]  \hfill (3)

where $r^+ = \sup \{r, 0\}$ and $r^- = |r| - r^+$. Define $T = \sup t$ where the supremum is taken over all partitions of $[a, b]$. If $T < \infty$, then $f$ is of bounded variation over $[a, b]$. We denote $f \in BV$.

Let $f : [a, b] \to \mathbb{R}$. Then $f$ is of bounded variation on $[a, b]$ if and only if $f$ is the difference between two monotone increasing real-valued functions on $[a, b]$. If $f$ is of bounded variation on $[a, b]$, then $f$ has countable discontinuities in $[a, b]$. Let $f \in BV[a, b]$. If

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\[ \|f\| = \lim_{x \to a^+} f(x) + T^b_a(f), \]  

then \((BV[a, b], \|\cdot\|)\) is a Banach space.

If \(f : [a, b] \to \mathbb{R}\) is of bounded variation on \([a, b]\), then \(f'(x)\) exists for almost all \(x\) in \([a, b]\). If \(f : [a, b] \to \mathbb{R}\) is integrable and \(F : [a, b] \to \mathbb{R}\) is defined by
\[ F(x) = \int_{[a,x]} f \, dm \text{ for } x \in [a, b], \]
then \(F\) is continuous and of bounded variation on \([a, b]\), and \(F'(x) = f(x)\) for almost all \(x \in [a, b]\).

If \(f : [a, b] \to \mathbb{R}\) is of bounded variation on \([a, b]\), then \(f\) can be written as \(f = g + h\) where \(g\) is an absolutely continuous function on \([a, b]\) and \(h\) is a singular function on \([a, b]\). A real-valued function defined on \([a, b]\) is absolutely continuous on \([a, b]\) if and only if it is continuous on \([a, b]\), is of bounded variation on \([a, b]\), and maps measurable sets onto measurable sets.

**RIEMANN-STIELTJES INTEGRAL**

Let \(f\) and \(\phi\) be two bounded functions defined on \([a, b]\) and \(\Gamma = \{a = x_0 < x_1 < \cdots < x_n = b\}\) be a partition of \([a, b]\). Choose \(\{\xi_i\}_{i=1}^n\) such that \(x_{i-1} \leq \xi_i \leq x_i\). Define
\[ RS(\Gamma, f, \phi) = \sum_{i=1}^n f(\xi_i)[\phi(x_i) - \phi(x_{i-1})]. \]

\(RS(\Gamma, f, \phi)\) is called the **Riemann-Stieltjes sum** for \(\Gamma\). If \(\lim_{|\Gamma| \to 0} RS(\Gamma, f, \phi)\) is finite, then we say the **Riemann-Stieltjes integral** of \(f\) with respect to \(\phi\) on \([a, b]\) exists and is denoted by
\[ \int_a^b f(x) \, d\phi(x) \quad \text{or} \quad \int_a^b f \, d\phi. \]

We write \(f \in RS(\phi)\).

If \(\int_a^b f \, d\phi\) exists, then \(\int_a^b \phi \, df\) exists and
\[ \int_a^b f \, d\phi = [f(b)\phi(b) - f(a)\phi(a)] - \int_a^b \phi \, df. \]

Let \(f\) be continuous on \([a, b]\) and \(\phi \in BV[a, b]\). Then \(\int_a^b \phi \, df\) exists, and
\[ \left| \int_a^b f \, d\phi \right| \leq (\sup_{[a,b]} |f|) \, T^b_a(\phi) \]
Now, let $\phi$ be increasing on $[a, b]$. If $\int_a^b f \, d\phi$ and $\int_a^b g \, d\phi$ exist, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, d\phi \leq \int_a^b g(x) \, d\phi.$$  \hfill (10)

$\int_a^b |f| \, d\phi$ exists as well with

$$\left| \int_a^b f(x) \, d\phi \right| \leq \int_a^b |f(x)| \, d\phi,$$  \hfill (11)

and $\int_a^b f(x)g(x) \, d\phi$ exists.

If $f$ is continuous on $[a, b]$ and $\phi \in BV[a, b]$, then $\int_a^b f \, d\phi$ exists. Similar to Riemann integration, by imposing some conditions on $f$ and $\phi$, the Mean Value Theorem is applicable to Riemann-Stieltjes integration.

Let $R = \{(x, y) | a \leq x \leq b, \ c \leq y \leq d\}$. Suppose $\phi \in BV[a, b]$, $\varphi \in BV[c, d]$ and $f$ is continuous on $R$. If $(x, y) \in R$, define

$$F(y) = \int_a^b f(x, y) \, d\phi, \quad G(x) = \int_c^d f(x, y) \, d\varphi.$$  \hfill (12)

Then both $\int_c^d F \, d\varphi$ and $\int_a^b G \, d\phi$ exist, and

$$\int_c^d F(y) \, d\varphi(y) = \int_a^b G(x) \, d\phi(x).$$  \hfill (13)

Note: (13) implies we can interchange the order of integration

$$\int_a^b \left[ \int_c^d f(x, y) \, d\varphi(y) \right] \, d\phi(x) = \int_c^d \left[ \int_a^b f(x, y) \, d\phi(x) \right] \, d\varphi(y).$$  \hfill (14)

**LITTLEWOOD-PALEY AND MULTIPLIER THEOREM**

$\phi$ is an $L^p(\mathbb{R})$ **multiplier** if there exists $k$ such that

$$\| U_\phi f \|_p \leq k \| f \|_p$$  \hfill (15)

for every $f$ in $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. We define the **multiplier norm** of $\phi$ as

$$\| \phi \|_{p, p} = \inf \{ k > 0 : \| U_\phi f \|_p \leq k \| f \|_p \}$$  \hfill (16)

and denote the set of $L^p(\mathbb{R})$ multipliers by $M_p$. 

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Finally, we will show the strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$. Let $(\Delta_i)_{i \in \mathbb{Z}}$ be the usual dyadic decomposition of $\mathbb{R}$. Suppose $\phi$ is a function on $\mathbb{R}$ such that

$$|\phi(y)| \leq c \text{ for all } y \in \mathbb{R}$$

(17)

and

$$\sup_i T_{\Delta_i}(\phi) \leq c,$$

(18)

Then $\phi \in M_p$ (the set of $L^p(\mathbb{R})$ multipliers) for all $p$ in $(1, \infty)$, and $\|\phi\|_{p,p} \leq cd_p$ where $d_p$ is a number depending only on $p$.

CONCLUSION

All the results in this project can be found from the references below. However, we have extracted, compiled and arranged the various results related to functions of bounded variations and its applications for the easy understanding of all readers.

REFERENCES